Estimating Projections of the Playable Set

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The playable set in differential game theory is the set of points in state space where a pursuer can guarantee capture of an evader under all possible control actions of the evader to the contrary. For problems of low dimension (one, two, and, perhaps, three dimensions), the playable set usually can be constructed using the necessary conditions provided by the playability max-min principle. This principle provides controls for both the pursuer and the evader that will drive the system along the boundary of the playable set. We show here that, for a number of higher-dimensional problems, the playability max-min principle can still be used to estimate the projection of a higher-dimensional playable set onto a lower dimension (usually one or two dimensions). This is done by finding both an overestimate and an underestimate for such a projection. This procedure is used to re-examine some previously published results on aircraft collision avoidance.

Introduction

CONSIDER a dynamical system of the form

$$\dot{x} = f(x,u,v) \tag{1}$$

where $x = [x_1,\ldots,x_n]^T$ is an n-dimensional state vector and $f = [f_1,\ldots,f_n]^T$ is a C^1 function of the state x and the scalar controls u and v. The dot denotes differentiation with respect to time. Assume that u is controlled by a pursuer and v is controlled by an evader. Both the pursuer's and evader's control can be any piecewise continuous function of time that satisfies the constraint conditions $u \in U$ and $v \in V$ with U and V defined by

$$U = \{u \in R | u_{\min} \le u \le u_{\max}\} \tag{2}$$

and

$$V = \{v \in R | v_{\min} \le v \le v_{\max}\} \tag{3}$$

The pursuer's objective is to drive the dynamical system to a connected compact target defined by

$$\theta(x) \ge 0 \tag{4}$$

and the evader's objective is to drive the dynamical system away from the same target.

A fundamental problem in differential game theory is to find those points in state space where the pursuer can guarantee to drive the system to the target under all possible control actions of the evader to the contrary. The set of all such points is called the playable set.[1] The boundary of the playable set is a barrier in the terminology of Isaacs,[2] and this boundary may be found for problems of low dimension by applying the playability max-min principle.[2,3] This principle is the game theoretic counterpart of the controllability maximum principle[3,4] used to find boundaries of controllable or reachable sets. The same dimensionality limitations apply as to the efficacy of these principles. They have been most successful for finding the boundaries of controllable, reachable, or playable sets for problems of one,[5] two,[1,5,6] and three dimensions.[7-9]

The playability max-min principle is a set of necessary conditions that must be satisfied by controls u and v in order to drive the system along the boundary of the playable set. In general, such controls can be found for a problem of any dimension; however, if the playable set is to be described by the trajectories that lie in it, then only lower-dimensional problems are practical. The playability max-min principle may be stated as follows: Let x be a point on the boundary of the playable set and let $u^* \in U$ and $v^* \in V$ be the value of the controls at this point that will maintain the system [Eq. (1)] along the boundary of the playable set. Then, there must exist a nonzero vector $\lambda = [\lambda_1,\ldots,\lambda_n]^T$ satisfying the adjoint equations

$$\dot{\lambda}^T = -\frac{\partial H}{\partial x} \tag{5}$$

such that the function H defined by

$$H(\lambda,x,u,v) = \lambda^T f(x,u,v) \tag{6}$$

satisfies the saddle point condition

$$H(\lambda,x,u,v^*) \le H(\lambda,x,u^*,v^*) \le H(\lambda,x,u^*,v) \tag{7}$$

for all $u \in U$ and $v \in V$ and every x on the boundary of the playable set. Furthermore,

$$H(\lambda,x,u^*,v^*) = 0 \tag{8}$$

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Trajectories that satisfy Eqs. (1–3) and (5–8) are obtained by integrating this system of equations backward in time from the target set defined by Eq. (4). If the boundary of the target set forms a part of the boundary of the playable set, then initial conditions (in the retro sense) for both x and λ are obtained from the requirement that λ be in the direction of the outward directed normal to the surface at this point.

Since use of the playability max-min principle to define the playable set is, for all practical purposes, limited to problems of three dimensions or less, it is of interest to consider its use in determining projections of higher-dimensional playable sets onto lower dimensions. We will use a method here similar to the one used by Vincent and Wu[10] to estimate projections of the controllable set.

Projected Controllable Sets

The method of Vincent and Wu[10] uses the controllability minimum principle to determine both upper and lower bounds for a projection of the controllable set. Since the ordering of the components of the state vector is arbitrary, assume that the first few components define the space onto which the controllable set is to be projected. In particular, let these components be designated by the vector

$$p = [x_1, \ldots, x_q] \tag{9}$$

where $1 \leq q \leq n-1$. Let

$$z = [z_{q+1}, \ldots, z_n] \tag{10}$$

be a vector related to x through a nonsingular linear transformation Q of the form

$$\begin{bmatrix} p \\ z \end{bmatrix} = \begin{bmatrix} I & 0 \\ Q_{21} & Q_{22} \end{bmatrix} [x] \tag{11}$$

If Q can be chosen such that the transformed state variables satisfy a system of equations of the form

$$\dot{z} = f_A(z,u) \tag{12}$$

$$w = g(z,u) \tag{13}$$

$$\dot{p} = f_C(p,w) \tag{14}$$

where w is a scalar "output" from Eq. (12) and f_A, g, and f_C are all C^1 functions of these arguments, then these equations may be used in various combinations to obtain both an underestimate and an overestimate of the projected controllable set. Note that Eqs. (12) and (13) are decoupled from Eq. (14), and that Eq. (14) expresses the dynamics of the projected system (e.g., $p = [x_1, x_2]^T$) in terms of the projected state variable p and an output w as obtained from Eq. (13). Equation (12) expresses the remaining dynamics as a function of the remaining transformed variables and the original input u.

The basic idea, supported by theorems to follow, is contained in Fig. 1. If the original system has an equivalent representation given by Eqs. (12–14), we may think of the input to system A as producing an output which, in turn, drives system C. First, we find a domain for the output w [see Eq. (15)]. If we then find the controllable set for system C with w as the input subject to the domain just obtained, the resulting controllable set must be an overestimate of the projection of the actual controllable set (Theorem 1). Since system C is of low dimension, we generally will be able to find a closed-loop control law that will drive system C along the boundaries of the controllable set just obtained. If this same control law is used to drive the input to system A, whose output in turn drives system C, then any region defined by the output of system C must be an underestimate of the projection of the actual controllable set (Theorem 2).

More specifically, if we let C_z denote the controllable set for system A subject to the control constraint $u \in U$ and we calculate a new constraint set W defined by

$$W = \{w \in R | w = g(z,w) \, \forall z \in C_z \text{ and } u \in U\} \tag{15}$$

then the following theorem[10] may be obtained:

Theorem 1: If $\mathrm{Proj}_p C_x$ is the projection of the controllable set for the original system onto p space and if C_p is the controllable set for system C subject to the control constraint $w \in W$, then

$$\mathrm{Proj}_p C_x \subseteq C_p \tag{16}$$

Provided that the dimension of z is reasonably small so that we can find C_z and, hence, W. Theorem 1 provides a convenient way of determining an overestimate for $\mathrm{Proj}_p C_x$. Theorem 2 (Ref. 10) provides a way for determining an underestimate for $\mathrm{Proj}_p C_x$:

Theorem 2: If $\mathrm{Proj}_p C_x$ is the projection of the controllable set for the original system onto p space and if $\mathrm{Proj}_p \tilde{D}$ is the projection of the domain of attraction onto p space for systems A and C together under the closed-loop control $\hat{u}(p)$ given by

$$\begin{aligned} \hat{u}(p) &= u_{\max}, && \text{if } \bar{w}(p) \geq u_{\max} \\ &= \bar{w}(p), && \text{if } u_{\min} \leq \bar{w}(p) \leq u_{\max} \\ &= u_{\min}, && \text{if } \bar{w}(p) \leq u_{\min} \end{aligned} \tag{17}$$

where $\bar{w}(p)$ is the feedback control law associated with maintaining system C on the boundary of C_p, then

$$\mathrm{Proj}_p \tilde{D} \subseteq \mathrm{Proj}_p C_x \tag{18}$$

As an example of how these theorems may be used, consider the third-order state-space representation of the unstable system

$$\dot{x}_1 = x_2 \tag{19}$$

$$\dot{x}_2 = x_3 \tag{20}$$

$$\dot{x}_3 = 6x_1 - 11x_2 + 6x_3 + u \tag{21}$$

with

$$|u| \leq 1 \tag{22}$$

Suppose that we were interested in the x_1, x_2 projection of the controllable set to the origin. Letting $y = x_1$ [which, in turn, implies $\dot{y} = x_2$ from Eq. (19)], one obtains the equivalent I/O representation

$$\dddot{y} - 6\ddot{y} + 11\dot{y} - 6y = u \tag{23}$$

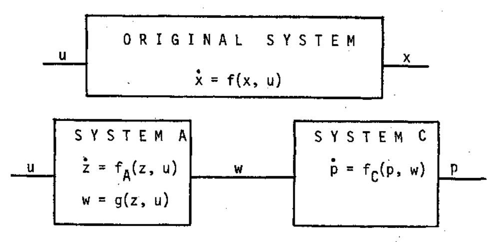

Fig. 1 Equivalent representation for a single-input control system.

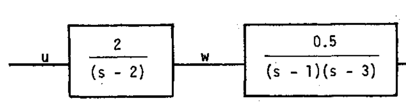

Fig. 2 Two-block representation of an I/O system.

with the transfer function

$$y(s) = \frac{1}{(s-1)(s-2)(s-3)} \tag{24}$$

Thus, this system may be broken easily into the two blocks shown in Fig. 2. The I/O system of the first block (system C) is given by

$$\ddot{y} - 4\dot{y} + 3y = w/2 \tag{25}$$

If we let $p_1 = y$ and $p_2 = \dot{y}$ (i.e., $p_1 = x_1$ and $p_2 = x_2$), Eq. (25) is equivalent to the state-space system

$$\dot{p}_1 = p_2 \tag{26}$$

and

$$\dot{p}_2 = -3p_1 + 4p_2 + (w/2) \tag{27}$$

while the I/O system of the second block (system A) has the state-space representation

$$\dot{z} = 2z + 2u \tag{28}$$

where

$$w = z \tag{29}$$

For this example, the transformation matrix $\boldsymbol{Q}$ is given by

$$\boldsymbol{Q} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 6 & -8 & 2 \end{bmatrix} \tag{30}$$

With this example in mind, and referring again to Fig. 1, we may see how both the upper and lower bounds for the projected controllable set may be obtained using Theorems 1 and 2. First, bounds on w are obtained from system A using Eq. (15). For the example, this is easy to do since w is equal to the single state variable for this system. In general, we obtain

$$w_{\min} \leq w \leq w_{\max} \tag{31}$$

where, for this case, $w_{\min} = -1.0$ and $w_{\max} = 1.0$. Now using w subject only to Eq. (31) as the input to system C, we find the control law for w, which yields trajectories on the boundary of the controllable set for system C. The resulting controllable set represents an overestimate for the actual projection of the true controllable set as provided by Theorem 1.

In order to obtain the underestimate, we apply the control law just determined for w as the input for u to drive both systems A and C together (i.e., the overall system). The resulting $\boldsymbol{p}$ state trajectories define an underestimate of the true controllable set as provided by Theorem 2.

The results of using this procedure on Eqs. (19–22) is illustrated in Fig. 3. Both the overestimate and the underestimate are obtained as noted. In this case, the underestimate is identical to the actual projection. This will always be true when the roots to the characteristic equation for the I/O system are real.[11]

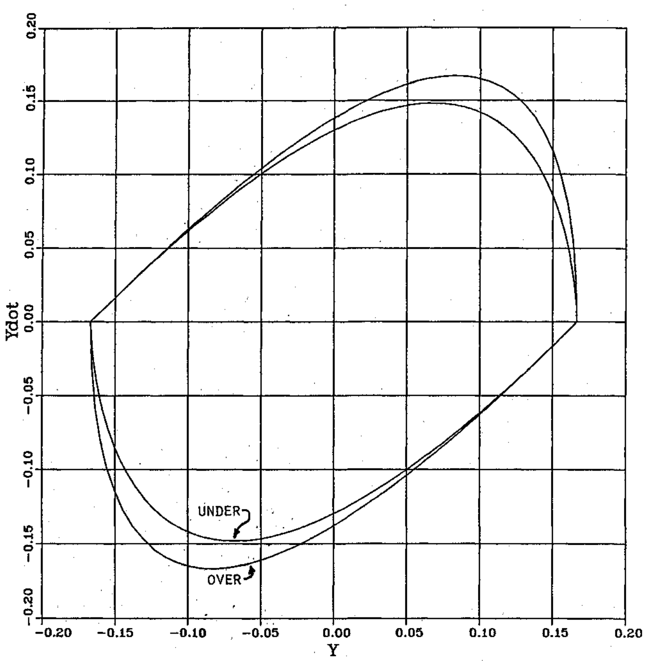

Fig. 3 Overestimate and underestimate for the projected controllable set to the origin.

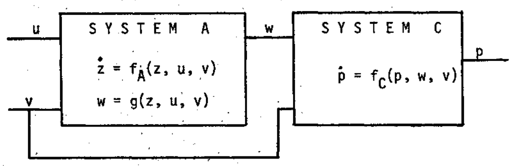

Fig. 4 Equivalent representation for a two-player game.

Projected Playable Set

In this case, the overall dynamics of the system are given in Eq. (1), with u and v bounded by Eqs. (2) and (3). A similar procedure can be used to find the upper and lower estimates for the projection of the pursuer's playable set, if as provided in Fig. 4, only the pursuer's control input is affected by the separation of the system into blocks. In Fig. 4, it is assumed that the v input to system C is unaffected by system A so that the projection of the playable set for the pursuer will be affected in exactly the same way as the controllable set was for the u controller of Fig. 1. In other words, Theorems 1 and 2 remain applicable to this situation with an appropriate change in terminology (i.e., replace "controllable" with "pursuer's playable").

The overall system is assumed to have an equivalent representation of the form

$$\dot{z} = f_A(z,u,v) \tag{32}$$

$$w = g(z,u,v) \tag{33}$$

$$\dot{p} = f_C(z,w,v) \tag{34}$$

with the same interpretation on the variables p, z, u, and w as before. Analogous to the controllable set case, we first find bounds on w using system A. Then, the pursuer's playable set for system C is found using w (now the pursuer's control) and v as inputs. This set is the overestimate for the pursuer's projected playable set. The underestimate for the pursuer's projected playable set is also found in a manner suggested by Theorem 2. If $\bar{w}(\boldsymbol{p})$ and $\bar{v}(\boldsymbol{p})$ are feedback control laws associated with maintaining system C on the boundary of the pursuer's playable set and if $\hat{u}(\boldsymbol{p})$ is obtained from Eq. (17), then using $\hat{u}(\boldsymbol{p})$ and $\bar{v}(\boldsymbol{p})$ to drive the overall system (systems A and C together) yields state trajectories in space which, in turn, define an underestimate for the projected playable set.

At this point, it is not clear how large a class of problems can be formulated in terms of Eqs. (12–14) for control problems or Eqs. (32–34) for game problems. The following example, however, does illustrate the applicability of the method to the well-known game of two cars.[2]

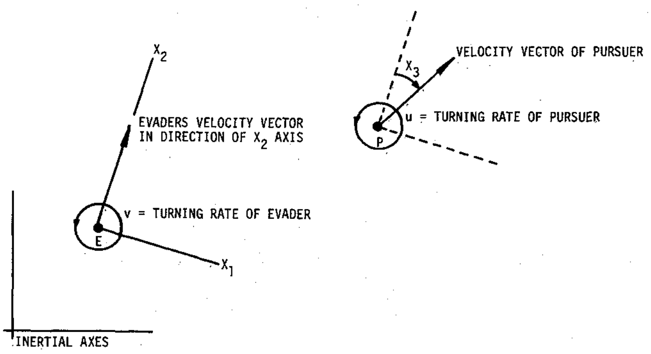

Fig. 5 Pursuer and evader moving in the horizontal plane.

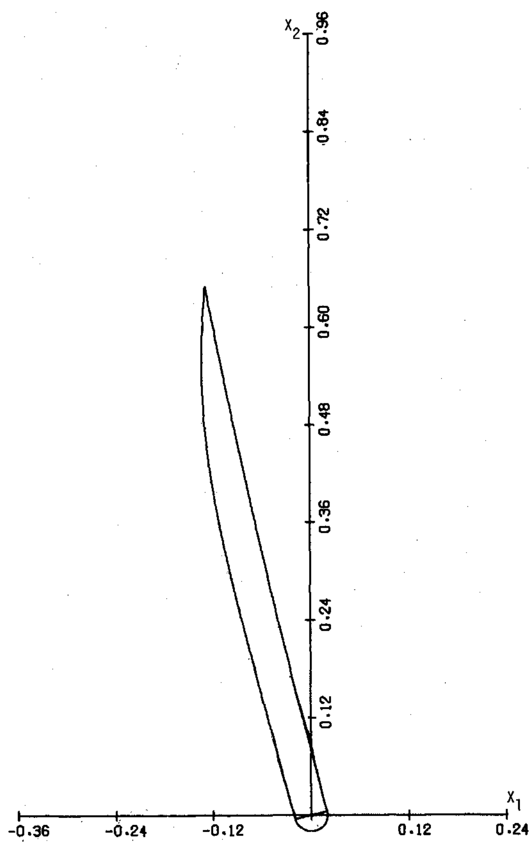

Fig. 6 Pursuer's playable set cross section of $x_3 = 135$ deg.

Aircraft Collision Avoidance

In an earlier work,[12] the problem of collision avoidance between two aircraft was studied in some detail using dynamics from the game of two cars.[2] Although both pilots wish to avoid a collision, in a worst-case situation, one pilot actually may use exactly the wrong maneuvers through negligence or lack of warning. We take the perspective of being on the aircraft actively trying to avoid a collision (the evader) in a worst-case situation. In particular, we wish to find the pursuer's playable set. Inside this set, a worst-case collision cannot be avoided. An evader-centered coordinate system is illustrated in Fig. 5. The pursuer is moving with a constant speed S_u with a turning rate u. The evader is moving with a constant speed S_v at a turning rate v. The x_1, x_2 coordinate system rotates so that x_2 is always aligned with the velocity vector of the evader and x_3 is the heading angle of the pursuer's velocity vector with respect to the x_2 axis. The dynamics of the overall system are given by

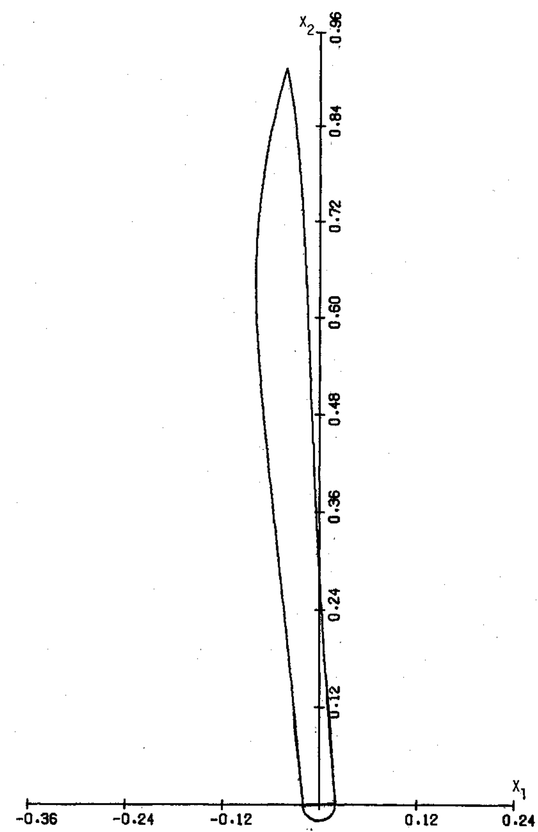

Fig. 7 Pursuer's playable set cross section at $x_3 = 165$ deg.

$$\dot{x}_1 = S_u \sin x_3 + v x_2 \tag{35}$$

$$\dot{x}_2 = S_u \cos x_3 - S_v - v x_1 \tag{36}$$

$$\dot{x}_3 = v - u \tag{37}$$

Using the necessary conditions [Eqs. (5–8)], the actual three-dimensional playable set can be determined. Figures 6 and 7 illustrate some cross sections of the playable set at specific values for x_3. These cross sections were obtained for parameter values $S_v = 1$, $S_u = 0.5$, $|v| \leq 1$, and $|u| \leq 2.5$, with the target set taken as an x_3 cylinder of radius $R = 0.02$. Enough cross sections were calculated in Vincent et al.[12] (with considerable computational effort!) to allow for a reasonable representation of the projection of the playable set onto the x_1,x_2 axis. This projection is illustrated in Fig. 8. A shortcut to obtaining this projection may now be obtained using the methods discussed previously.

Equations (35–37) may be represented as a system of the form of Eqs. (32–34) by letting $x_1 = p_1$, $x_2 = p_2$, and $x_3 = w = z$. We obtain for system C,

$$\dot{p}_1 = S_u \sin w + v p_2 \tag{38}$$

$$\dot{p}_2 = S_u \cos w - S_v - v p_1 \tag{39}$$

and for system A,

$$\dot{z} = v - u \tag{40}$$

$$w = z \tag{41}$$

We note from Eq. (40) that, even with bounds on u and v, the playable set for system A will be unbounded; hence, w is unbounded. We first obtain an overestimate of the projected playable set by finding the playable set for Eqs. (38) and (39) with w unbounded and v bounded by $|v| \leq 1$. From the necessary conditions [Eqs. (5–8)], the control for v is obtained from

$$v = +1, \quad \text{if } \sigma_v > 0$$
$$= -1, \quad \text{if } \sigma_v < 0$$
$$= 0, \quad \text{if } \sigma_v \equiv 0 \qquad (42)$$

where σ_v is the switching function

$$\sigma_v = \lambda_1 x_2 - \lambda_2 x_1 \qquad (43)$$

and the control for w is obtained from

$$\tan w = \lambda_1/\lambda_2 \qquad (44)$$

The adjoint variables are obtained from

$$\dot{\lambda}_1 = v\lambda_2 \qquad (45)$$

and

$$\dot{\lambda}_2 = -v\lambda_1 \qquad (46)$$

with initial conditions for both $x(0)$ and $\lambda(0)$ obtained from the requirements that $H = 0$ and that λ be in the direction of the outward directed normal to the circle

$$x_1^2 + x_2^2 \leq R^2 \qquad (47)$$

at the point where the playable set is just tangent to the circle. The result of integrating Eqs. (38) and (39) subject to Eqs. (42–46) is illustrated as the outer curve in Fig. 9. This overestimate is just slightly larger than the actual projection.

The underestimate is obtained by using the same control law for u as was used for w except that, if this control law calls for $|u| > 2.5$, it must be replaced by the appropriate saturation limit. This time, integrating Eqs. (38) and (39) subject to Eqs. (42–46) adjusted for the saturation limit on u, we obtain the inner curve given in Fig. 9. This underestimate is just slightly smaller than the actual projection.

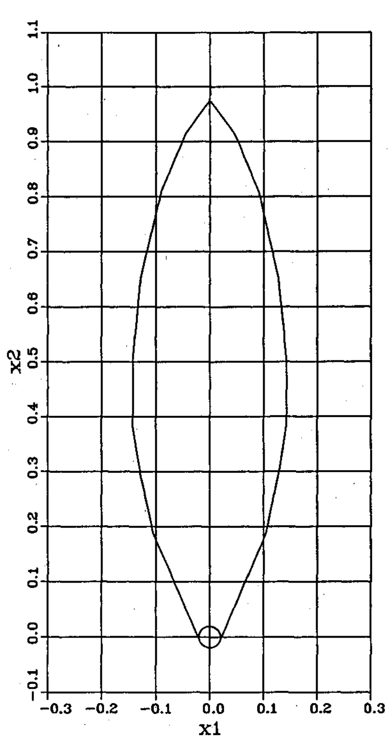

Fig. 8 Projection of the pursuer's playable set.

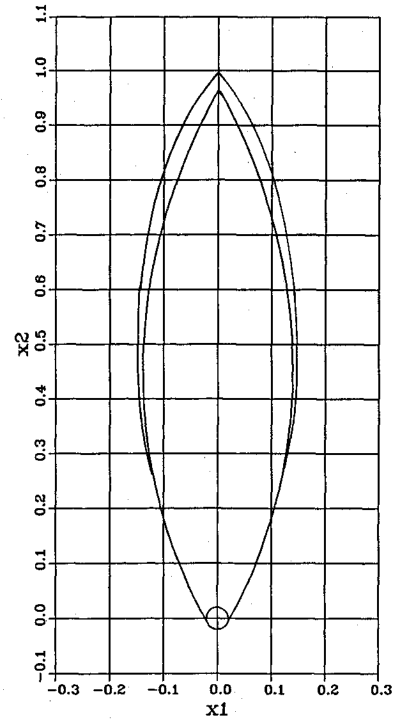

Fig. 9 Underestimate and overestimate of the projection of the pursuer's playable set.

Discussion

Clearly, we can obtain a very good estimate for the actual projection of the playable set, as is evident from Figs. 8 and 9. This result is rather surprising in view of the fact that the u control used in the underestimate would seem to be approximate at best. Moreover, the necessary conditions [Eqs. (5–8)] applied to the original system [Eqs. (35–37)] require that both u and v be bang-bang except along a singular arc.

Even if the results were not this accurate, the computational advantage of the estimating procedure used here can be appreciated only if one has attempted to solve problems of this type both ways. For higher-dimensional problems, one may have no choice except to use an estimating procedure. Other procedures are available[13,14] for estimating controllable sets based on Lyapunov methods, and they may no doubt be applied to the game situation as well. Since the Lyapunov methods always yield underestimates, there is some disadvantage to this approach. With the approach presented here, both an underestimate and an overestimate are obtained. The disadvantage is that it may not always be possible to reformulate the system as Eqs. (12–14) for control problems or Eqs. (32–34) for game problems. However, such a reformulation is always possible for a large class of linear control problems.[10]

Of course, it is always useful to have a complete description of the playable set. In the collision avoidance example given here, the evader would know that, if the target lies outside this set, he can always execute a maneuver and avoid a collision. Of what use then is the projection information? Consider the situation in which the heading angle of the target is unknown or not known reliably. In this case, if the target lies outside the projection of the playable set, the evader would still have the same fail-safe information, albeit in reference to a set larger than might be necessary.

Conclusions

A method originally developed for finding over- and underestimates of projections of the controllable set is also applicable for finding over- and underestimates for the playable set provided that the original system dynamics can be re-expressed in a particular decoupled form. This method, applied to the three-dimensional game of two cars, gives over- and underestimates that are very close to the actual projection of the playable set onto the two-dimensional space consisting of the relative position coordinates of the pursuer with respect to the evader. The computational effort needed to obtain this result using the method presented is trivial in comparison with the computational effort required to find the actual projection for the full three-dimensional problem.

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